Module 10: Optimization

Learning goal for Module 10

Much of economics focuses on maximizing or minimizing some quantity (revenue, costs, profits, etc.), or finding an equilibrium point. Excel supports a range of optimization problems common to economic and business analytics. This Module will equip you with a basic understanding of these techniques, which lay the foundation for linear programming, a common starting point for the study of operations research. We start with the simple economics of demand, then add supply, and then look at solving basic optimization problems.

By the end of this Module, will :

- Recognize an optimization problem
- Translate an optimization problem into a formal structure in Excel
- Understand the use of Solver

Caution: Students find this and the next two Modules quite challenging. Be prepared to spend extra time studying the problems and examples.

1. Introduction

Optimization problems occur throughout economic and business decision-making. What change in price will maximize revenue? How do I combine workers, equipment, and energy to produce the least unit cost? What will a tax do to demand? How will rationing reduce consumer welfare?

This Module examines consumer demand, elasticity, and the dynamics of demand and supply using Solver, a most important Excel tool.

1.1. Consumer demand

A main element of microeconomics is the analysis of consumer demand. Why are people willing to exchange resources (time, money) to acquire goods and services? Our desire/need for something, that is our quantity demanded (Q_d), starts with survival, and once we have met that goal, psychological and social influences shape our “demands.” The actual quantities of any goods or services we consume depends on many determinants (causes, drivers…), the most important of which include:

- ***Own price*** - the price of the product/service
- ***Cross prices*** - the price(s) of competing/complementary products/services
- ***Income*** - current and expected
- ***Wealth*** - especially when used to “leverage” (serve as collateral) for a loan, such as a mortgage
- ***Interest rates*** - to support financing of that loan

The *own price* (P_x) is the most important determinant of *quantity demanded* of good X; for most situations, an inverse relationship exists between P_X and Q_d. This results in the familiar demand "curve" shown in **Error! Reference source not found.**. A price increase results is less quantity demanded and vice versa.

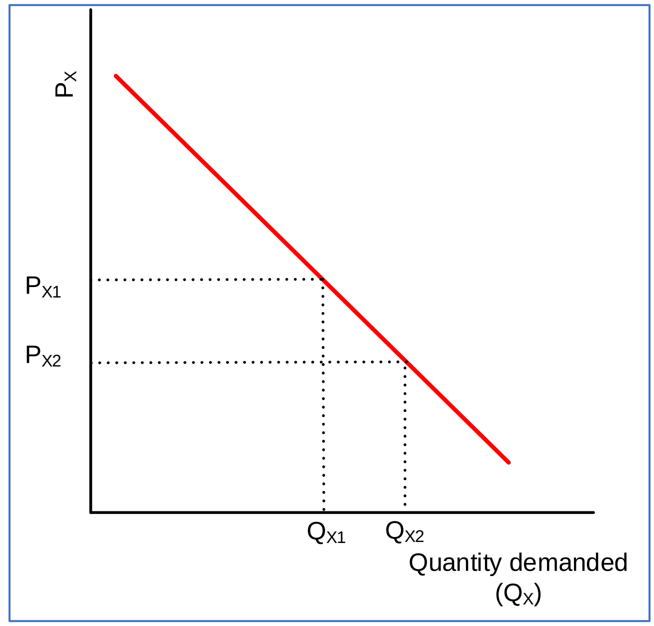


Figure 1: Own price demand

Cross prices, the price of other goods Y and Z, can be more complex. If consumers can/will substitute (movement along the demand relationship) Y for X (chicken for beef), then as the price of Y (PY) increases, the quantity demanded of X will increase. If Y and X are complements, car and gasoline, then as PY increases, QX decreases. Figure 3 shows these relationships. Finally, the relation between income and QX appears in Figure 3.

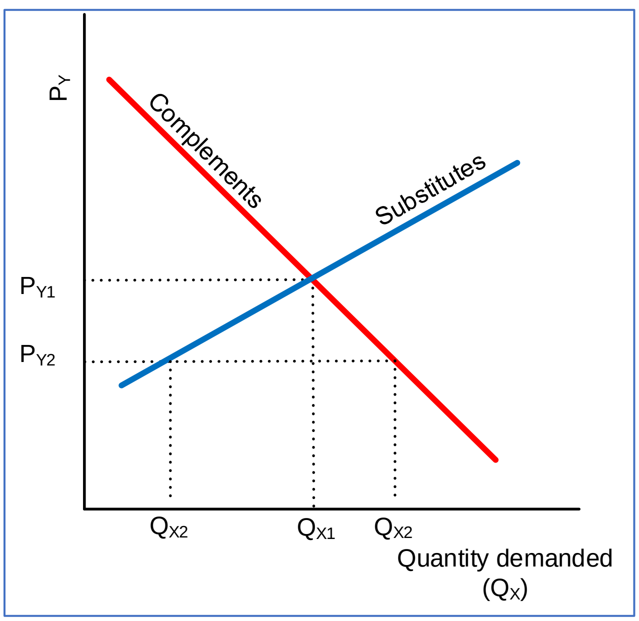


Figure 2: Substitutes and complements

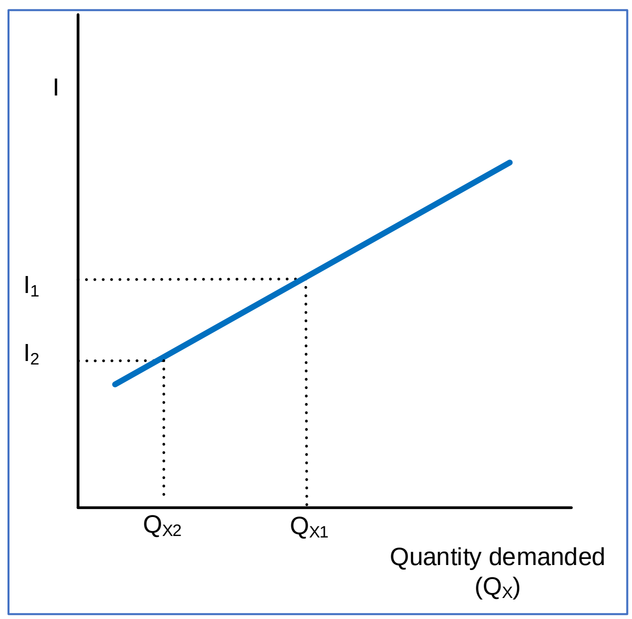


Figure 3: Income and Qx

When we combine two determinants of demand on a single chart, such as own price and income, the influence of the "secondary" dimension appears as a shift in the primary relationship, as seen in Figure 4. Note that demand analysis usually assumes that own price is the primary determinant of demand. Here an increase in income shifts the demand relationship. At P_{X1}, an income increase raises quantity demanded from Q_{X1} to Q_{X2}.

Using geometry to portray a demand relation with many determinants is "clunky." The better way is to write a demand relation as appeared in Module 6.

$$Q_{DX} = a_0 + a_1 P_X + a_2 P_Y + a_3 I$$

Note that Q_{DX} denotes the quantity demanded for X. No error term appears and neither do the subscripts i or t since this is an abstract equation and not an estimator for data.

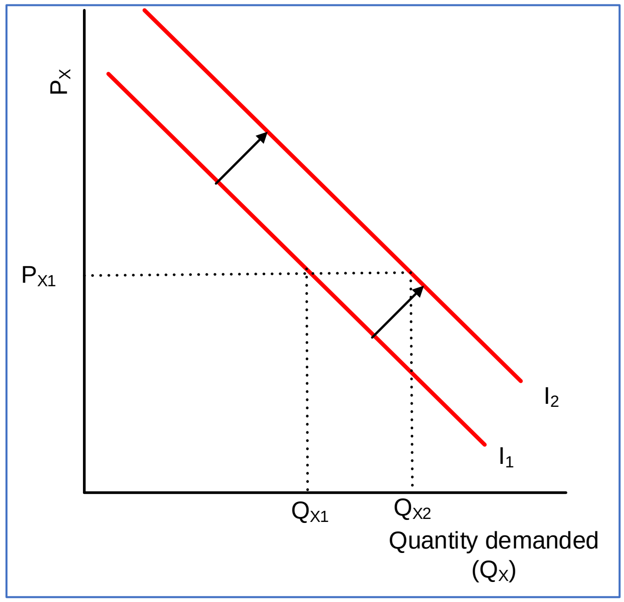


Figure 4: Price and income affect demand

1.2. Own price elasticity

The own price elasticity of demand is often an important issue for public policy and market studies. A firm wishes to offer a good or service that has an inelastic own price elasticity because this offers the firm some ability to raise prices and increase revenue. Advertising and celebrity influencers on social media are common ways to increase brand

Own price elasticity of demand:

The percentage change in the quantity of a good demanded resulting from a 1% change in its own price.

Elasticity varies along the demand curve or relation

- $\varepsilon > 1$ (demand is elastic – quantity demanded responsive to price)
- $\varepsilon < 1$ (demand is inelastic – quantity demanded is not responsive to price)
- $\varepsilon = 1$

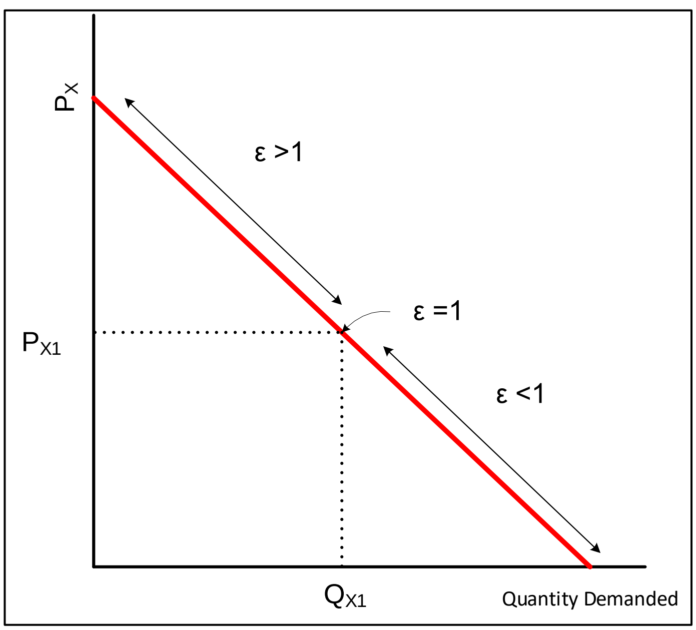


Figure 5: Elasticity varies along the demand curve

equity, which reduces the responsiveness of total revenue to price. Public policy analysts use demand elasticity to assess the degree of market power possessed by a firm's goods and services.

Three special cases

- **Perfectly Elastic** demand
  - Price elasticity of demand is infinite.
  - Even the slightest change in price leads consumers to find substitutes. (All fast food is equally disgusting? A one cent increase in the price of the Big Mac stampedes people over to A&W.)
- **Perfectly Inelastic** demand
  - Price elasticity of demand is zero.
  - Consumers do not switch to substitutes even when price increases dramatically. (My only phone will ever be an iPhone regardless of price!)
- **Unit Elastic** demand
  - Regardless of the price selected, total expenditure is unchanged.
  - P = K/Q where K≠0 (Or P*Q=K)
  - This is a mathematical curiosity of interest only to economic gearheads.

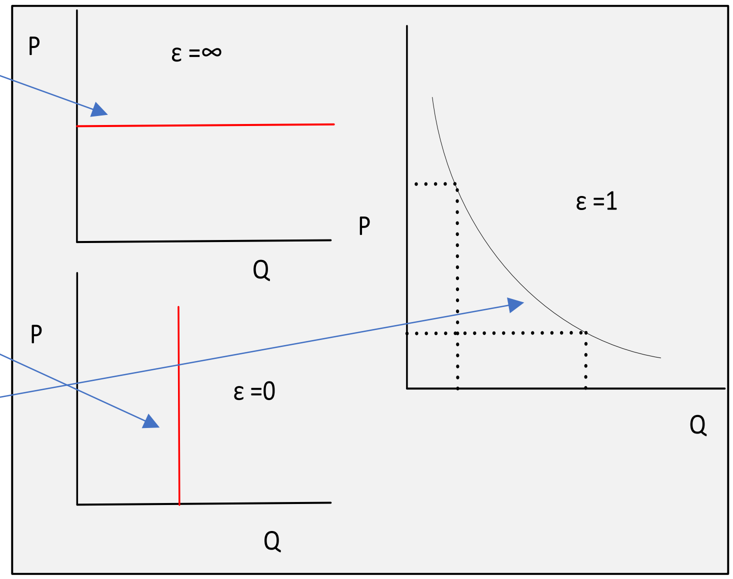


Figure 6: Examples of special cases

Price elastic products/services: Quantity demanded is highly responsive to price changes. These are "desirables."

ε>1	P↑ R↓	P↓ R↑
ε=1	P↑ R→	P↓ R→
ε<1	P↑ R↑	P↓ R↓

Price inelastic products/services: Quantity demanded is not responsive to price changes. These are "necessities" and include additive goods, such as tobacco, hard drugs, and services such as porn.

Examples

- ***Inelastic demand***: Apple iPhone(s) (Although demand has been weakening in the face of competition from Samsung). Any business wants to reduce the price elasticity through advertising, creating affinity cards, introducing new features to create "brand loyalty."

- *Elastic demand*: Fast food (For busy families and students with little time for food shopping and cooking, fast food may be a necessity, but the fast food at a particular restaurant may be elastic). Certain commodities, such as aluminum, sold internationally are usually highly elastic. Minor variation exists in the product itself.

The figure presents some typical elasticity measures. In each case, try to imagine what causes the level observed. Why are green peas highly price elastic while opera tickets are inelastic? What's the price elasticity of heroin or crack cocaine to an addict?

Graphing Demand.xlsx

A 1% increase in price leads to a 2.8% decrease in quantity demanded of peas.

A 1% increase in price leads to a .18% decrease in quantity demanded for opera tickets.

Total Revenue vs Elasticity.xlsx

Good or service	Price elasticity
Green peas	2.80
Restaurant meals	1.63
Automobiles	1.35
Electricity	1.20
Beer	1.19
Movies	0.87
Air travel (foreign)	0.77
Shoes	0.70
Coffee	0.25
Theatre, opera	0.18

Figure 7: Typical examples of own price elasticities

Video: Total Revenue vs Elasticity 1

Video: Total Revenue vs Elasticity 2

1.3. Modelling a demand relationship in Excel

The general equation for consumer demand appears as

$$Q_{DX} = a_0 + a_1 P_X + a_2 P_Y + a_3 I + a_4 r + a_5 W$$

where Q_{DX} is the quantity demanded of X, P_X and P_Y the prices of X and Y (either a complement or substitute), *I* is income, *r* the interest rate, and *W* a measure of wealth. The a_i are coefficients that show the change in Q_{Dx} for a unit change in the respective independent variable. Recall from Module 6 that these coefficients were the regression estimates. The expected sign for a1 < 0 (which is why own price elasticity is negative), while a2 may be positive or negative, a3 has an expected sign > 1, as does a5. The coefficient on interest – r – is expected to be negative.

For portraying simple demand and supply relationships in Excel, see this example.

Graphing Demand and Supply.xlsx

The first challenge is to "draw" a demand relationship. Given a demand equation and schedule of prices, it is a simple task to create a demand table then graph it as a scatter diagram.

A specific model appears as

$$Q_{DX} = 1200 - .25P_X + .3P_Y + 3.5I - 1100r + 2.7W$$

This example states that the quantity demanded is 1200 units, minus .25 times the price of the product, plus .3 times the price of a substitute, plus 3.5 times income, minus 1100 times the interest rate, plus 2.7 times the wealth. Also, this is a linear additive model (all the independent variables are of order 1, and their effect on price is additive). A non-linear additive model might appear as

$$Q_{DX} = a_0 + a_1P_X + a_2P_X^2 + a_3P_Y + a_4r + a_5I^{.5} + a_6\ln W$$

A multiplicative model might be

$$Q_{DX} = a_0P_X^{a_1}P_Y^{a_2}I^{a_3}W^{a_4}$$

Excel can explore the relationships between prices, income, and other independent variables determining income. For a slick way to change the values in a dashboard, watch these videos.

1.4. Estimating demand elasticity (own price) – Traditional view

Own price elasticity is the percentage change in the quantity of a good demanded, resulting from a 1% change in its price. More formally, price elasticity = % change in quantity demanded / % change in price, or

$$\varepsilon = \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta P}{P}\right)$$

Since the slope of the demand curve is $\frac{\Delta P}{\Delta Q}$, another way to express it is:

$$\varepsilon = (P / Q)(1 / slope)$$

The own price elasticity is only partly determined by the slope.

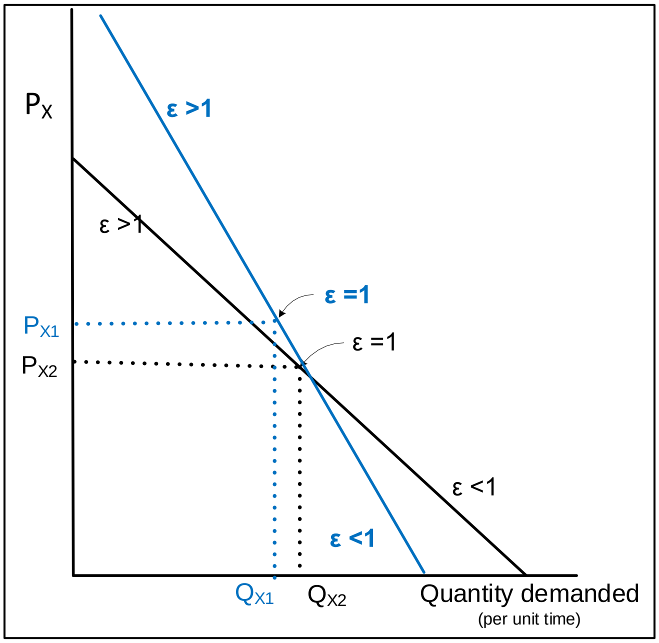


Figure 8: Elasticity varies along the demand curve

A common error is to associate elasticity with the slope of the demand; that is only part of the story. Elasticity depends on the slope of the demand ***and*** the position on the curve. More consequential for business decisions is the relation between elasticity and total revenue.

See example [**Total Revenue vs Elasticity.xlsx**].

This relationship appears in Figure 10. Starting in the upper left, the shaded rectangle *obca* shows the total revenue resulting from a price quantity combination. As price rises, quantity, and total revenue fall. At some price, nothing sells. For a price of $0, everything sells, but total revenue is also zero. The total revenue graph shows this. Figure 10 shows this in the price and quantity dimensions.

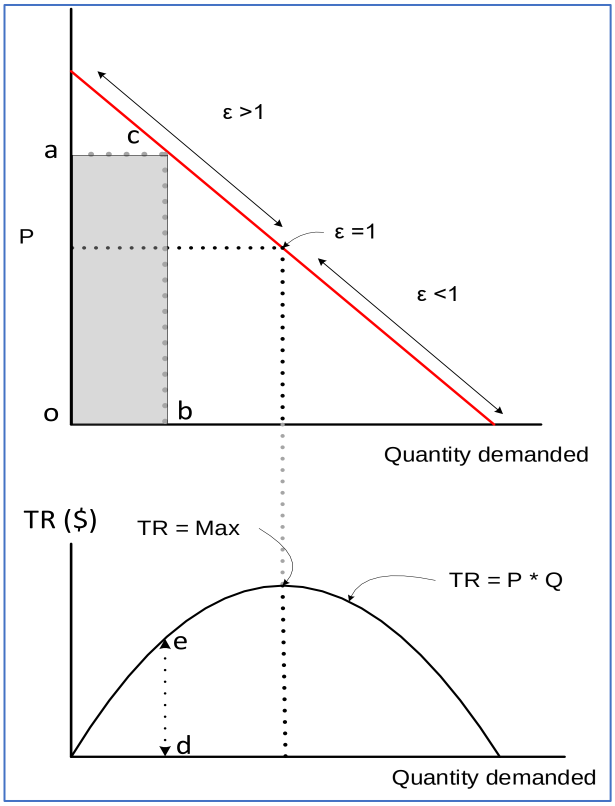


Figure 9: Elasticity and total revenue

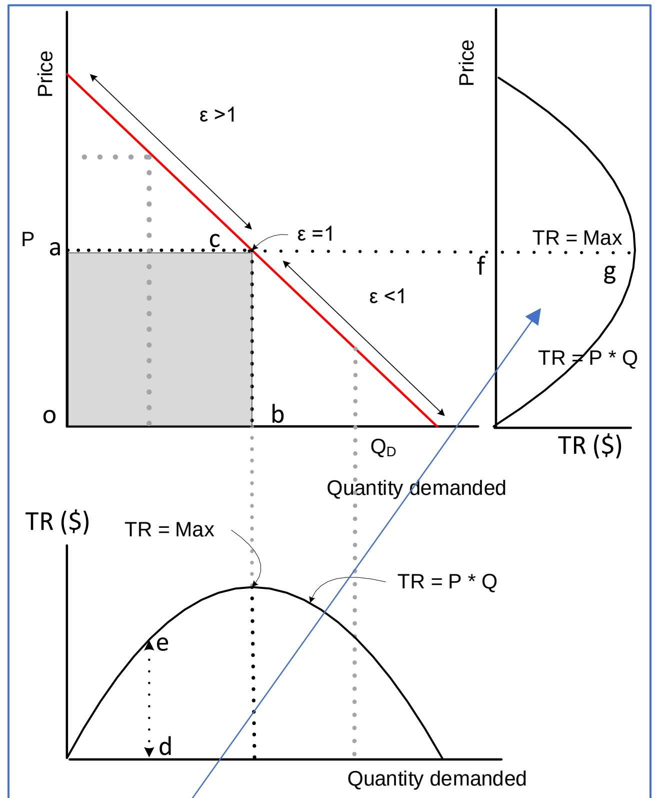


Figure 10: Elasticity and total revenue – two dimensions

> It is possible to view the price, quantity, and TR relationship in two dimensions
>
> The price vs TR relationship is the better way to show this since we assume that price is the independent variable.

1.5. Own price elasticity ("modern" view)

Alfred Marshall did much to develop ideas of demand, but he also made a curious choice in how to show price and quantity. In consumer demand theory, price is an independent variable, to use the language of Module 6. In graphical presentations of any causal theory, expressed as Y = f(X), normally the horizontal axis has the designation as the "X-axis." The dependent variable Y appears on the vertical axis.

When modelling demand (and supply) using Excel, it is clearer to maintain the standard mathematical practice of having the dependent variable (QD) quantity represented on the vertical axis and price on the horizontal axis. This changes the position of the elasticity along the demand curve and the calculation for elasticity must "invert."

The price – demand function and its elasticity now appear as:

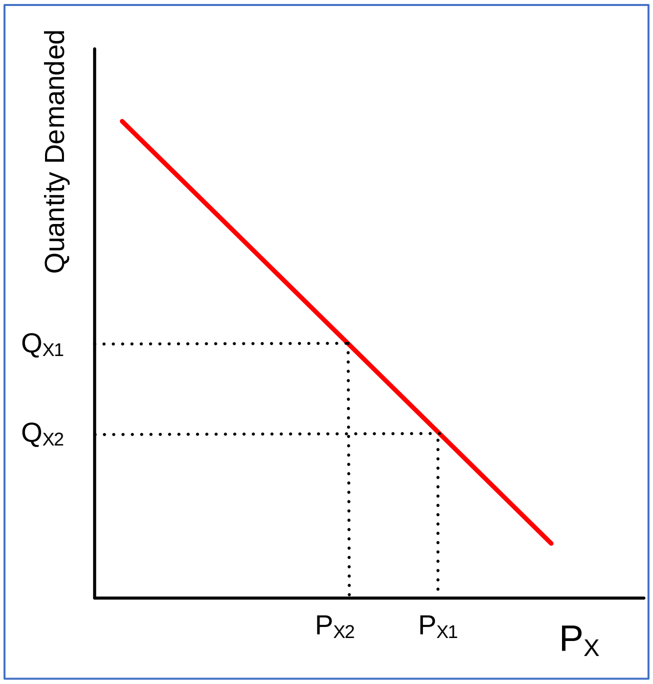


Figure 11: Demand with the "correct" axes

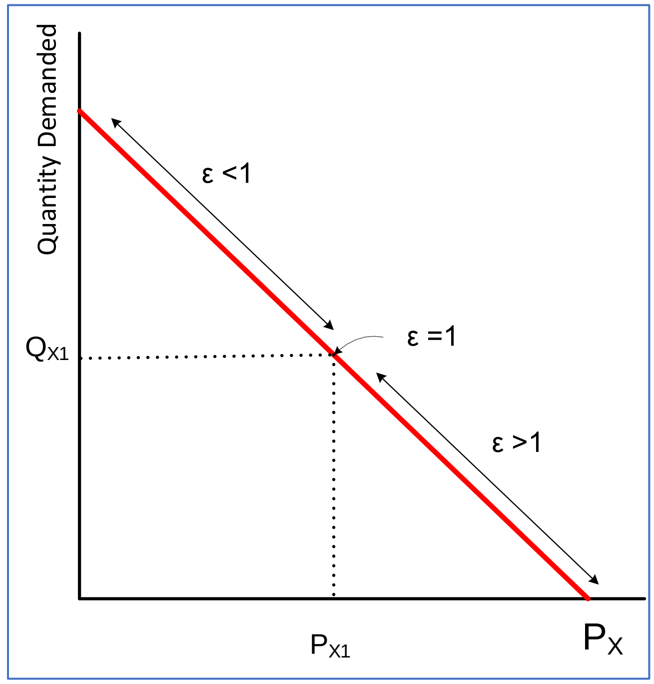


Figure 12: Elasticity – revised

1.6. Income and cross-price elasticity

Think of a demand function:

$$Q_{DX} = a_0 + a_1 P_x + a_2 I$$

where I is income and a_2 is usually positive, unlike a_1, which is negative.

The *income elasticity* is the "percentage change in quantity demanded for a unit percent change in income," or

$$\varepsilon_I = \frac{\Delta Q_{DX} / Q_{DX}}{\Delta I / I}$$

Normal good: a good with a positive income elasticity of demand ($\varepsilon_y > 0$ but $\varepsilon_y < 1$)

Inferior good: a good with a negative income elasticity of demand ($\varepsilon_y < 0$)

Luxury good: a good with an income elasticity of demand greater than one; a subset of normal good ($\varepsilon_y > 1$)

Using the demand relation

$$Q_{DX} = a_0 + a_1 P_X + a_3 P_Y + a_3 I$$

the cross-price elasticity of Q_{DX} to P_Y is

$$\varepsilon_{XY} = \frac{\Delta Q_X / Q_X}{\Delta P_Y / P_Y}$$

Substitutes: two goods whose cross-price elasticity is positive (price increase in good 1 produces a quantity demand increase in good 2, and vice versa)

Complements: two goods whose cross-price elasticity is negative (price increase in good 1 produces a quantity demand decrease in good 2, and vice versa)

See **Exercise 10.3** and its solution to learn about calculating elasticities in Excel.

> Work through the exercises and examples to understand the process of calculating elasticities in Excel.

1.7. A note on supply

In microeconomics, quantity supplied of any product has determinants, such as

- ***Own price*** – price of the product (PX), the same as demand
- ***Costs of production*** *– denoted generically as C, but itself determined by categories such as fixed and marginal costs*
- ***Interest rates*** *– technically part of C and denoted as "r," but often identified separately for certain sectors, such as housing*

The supply equation appears as

$$Q_{SX} = b_0 + b_1 P_X + b_2 C + b_3 r\,,$$

where the expected sign on b1 > 0, b2 < 0, and b3 < 0.

1.8. The demand-supply system

The demand-supply equations usually appear as part of a system of two equations and an identity.

$$Q_{DX} = a_0 + a_1 P_X + a_2 P_Y + a_3 I + a_4 r + a_5 W$$
$$Q_{SX} = b_0 + b_1 P_X + b_2 C + b_3 r$$
$$Q_{DX} = Q_{SX}$$

To show key ideas, we may abbreviate the system to

$$Q_{DX} = a_0 + a_1 P_X + a_2 I$$
$$Q_{SX} = b_0 + b_1 P_X + b_2 r$$
$$Q_{DX} = Q_{SX}$$

Note the renumbering of the coefficients.

The third equation seems unusual. The demand and supply equations are *behavioural* since they describe/predict consumer/producer reaction or behaviour in response to changes in one or more independent variables. The third equation is different – it is an identity. Identity expressions are mathematical "enforcers" that direct the solution when something changes in the behaviour equations.

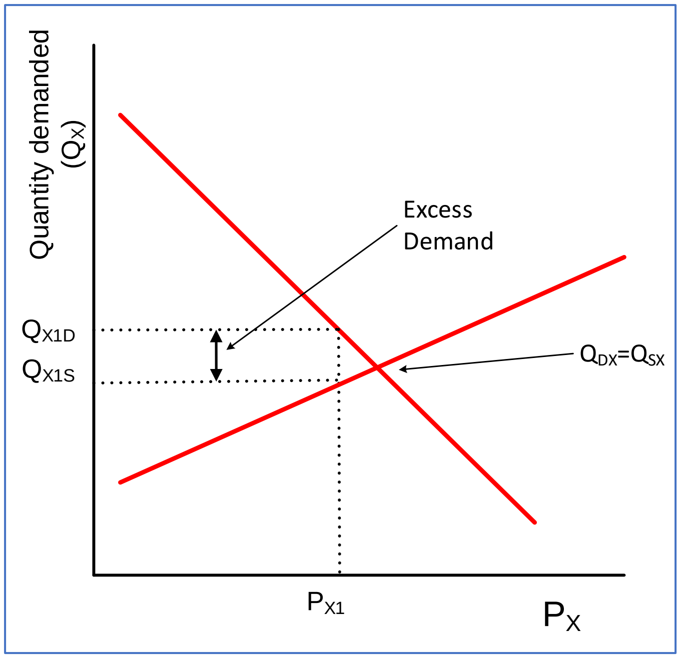


Figure 13: The role of the identity

Imagine one were at any point on the demand or supply curves as shown in Figure 13, say (P_{X1},Q_{X1}). At this point, demand exceeds supply and consumers will bid prices up, reducing the quantity demanded and increasing the quantity supplied. (Since we have reversed the axes, this may take a moment to puzzle out.) The presence of excess demand results in empty store shelves and concert tickets selling out in minutes. Suppliers react by increasing prices, which results in reduced demand and increased supply, stopping at the point where Q_{DX}=Q_{SX}. The reverse occurs with excess supply.

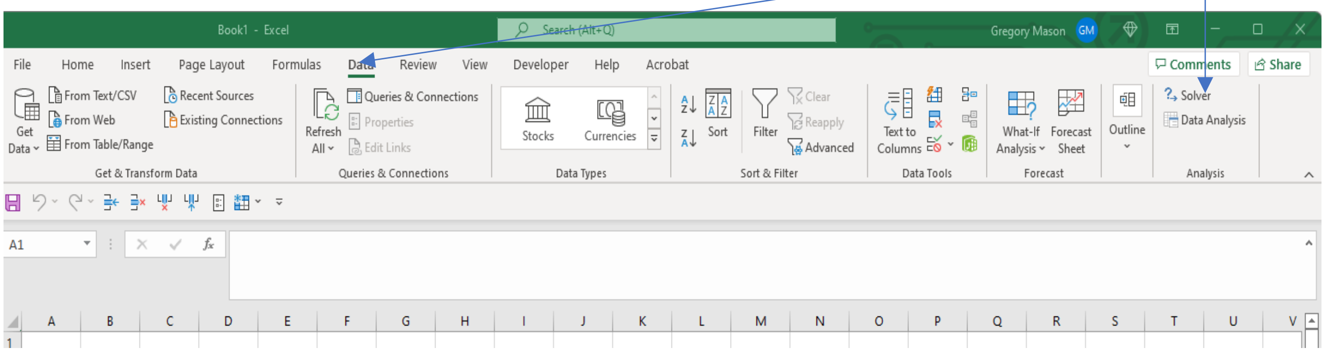
Finally, note that these two expressions are equivalent:

$$Q_{SX} = Q_{DX}$$
$$Q_{SX} - Q_{DX} = 0$$

2. Using Solver

Excel has some powerful modelling procedures, among the most useful being Solver. This is a "constrained optimizing algorithm," which simply means finding the optimum when resource constraints exist. For example, a student wishes to maximize their GPA for several courses. This means allocating study time, subject to the constraint that a week has 168 hours with a certain amount of sleep and other biological maintenance required. Solver offers a method for finding how to allocate limited resources to optimize the goals subject to constraints. In this context, optimization can mean maximization of GPA or finding the number of hours to allocate to each course to reach a GPA of 3.4 exactly.

The first step is to ensure you have Solver activated in Excel 365. It appears under the data tab.



If this does not appear, go to Files →Options → Add-ins and activate Solver.

2.1. The objective function

A key skill in finding the solution to optimization problems is to understand the goal or object of the optimization. This means defining the objective and then writing a formula known as the objective function. Many students find it new and challenging. It is one thing to say a consumer maximizes utility or an organization minimizes cost, but it can be quite difficult to express this in a formula.

Optimization means to find a maximum/minimum/equilibrium value according to some rule.

- *Example:* Assuming you know the demand relationship, what price will maximize total revenue?
- *Example:* If government places a 10% tax on cannabis gummies, what is the new equilibrium price and quantity?
- *Example:* You are the economist for a city government and the city council has asked you to calculate what increase in the water rates will raise a target value of revenue.
- *Example:* You are a major cell network provider and the system crashes for two days. What compensation in the form of a cash payment will minimize the number of

customers that switch to another network? (This is a tough one and requires one to calculate the marginal reduction in customer exit for a dollar value in compensation.)

Key steps in problem definition are:

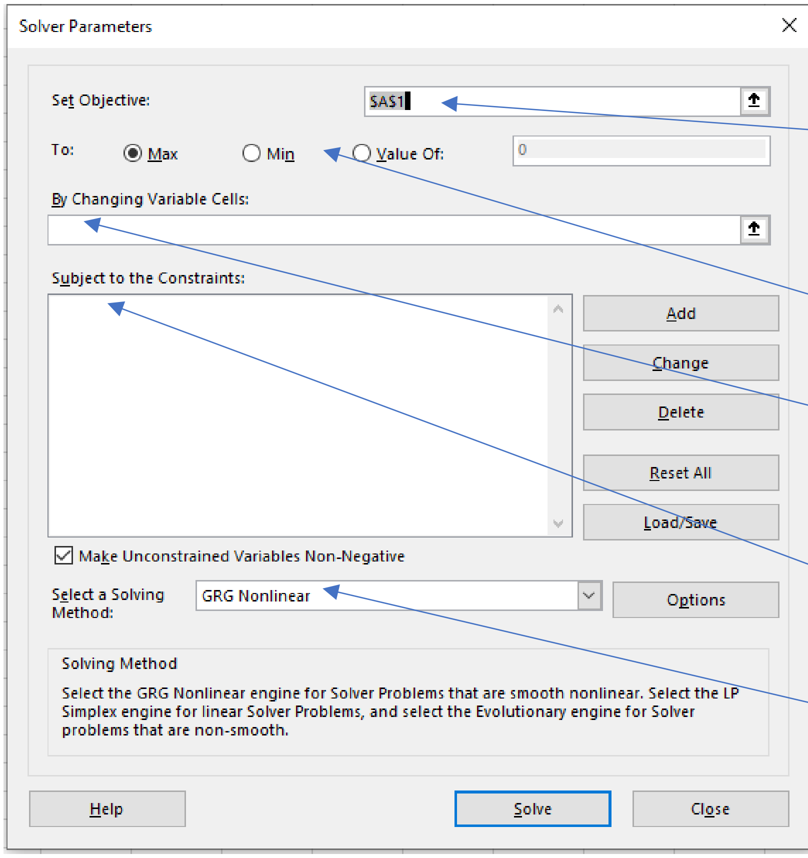
- Specify the objective in plain language. Trigger words such as *minimize*, *maximize*, and *new equilibrium* are important.
- Define the behavioural equations in Excel. Note: so far in this Module the behavioural equations are demand and supply; but each problem will have its own context.
- Define the optimization problem using an equation known as the *objective function* and express it as an equation in a cell.
- Decide whether you are maximizing, minimizing, or finding an equality.
- Determine the action variables that Solver will adjust to find the optimal value.
- Define the constraints on the action variables.

2.2. Optimization: General setup

The general setup will appear as

$$\text{Max/Min } F = f(y, x_i)$$
$$\text{ST } g(x_i) = a$$

This will seem very abstract but working through problems is the best way to become comfortable with the process. First, consider the Solver setup process for finding the equilibrium price and quantity for a simple demand and supply model. The general interface appears as follows:



Setting up a demand and supply problem

1. Set Objective is a cell that defines a formula. It can refer only to a single cell that has the formula for the maximum/minimum problem. This cannot be a range.
2. Set whether the objective is to maximize, minimize, or meet an exact condition.
3. Then define the cell(s) with the parameters/variables that are being varied. This can be a range.
4. Finally, many problems have constraints. (We will use this in Module 11.)
5. This dictates the type of solution process – for now use GRG Nonlinear.

Solver works for a range of optimization problems. Consider this example:

- *Example*: You are managing a Mexican resort hotel with a high and low season, implying two different demand relationships. You wish to find a price common to both seasons that maximizes revenue. One wrinkle is that your hotel only has 5,000 room-days in total (A "room-day" is one 24 hour period when a patron can rent a room. The total number of rooms time the days the resort is open, is the total room days and measures capacity.)

Hola Amigos.xlsx

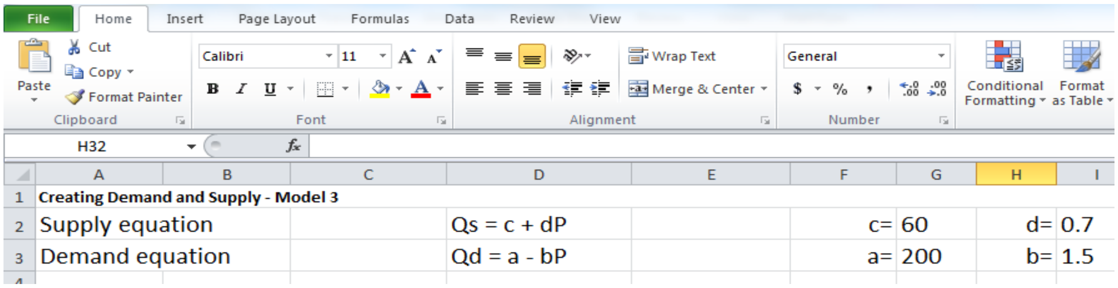
Video: Hola Amigos

2.3. Demand and supply: Equilibrium as the objective

The simple demand and supply model appears as

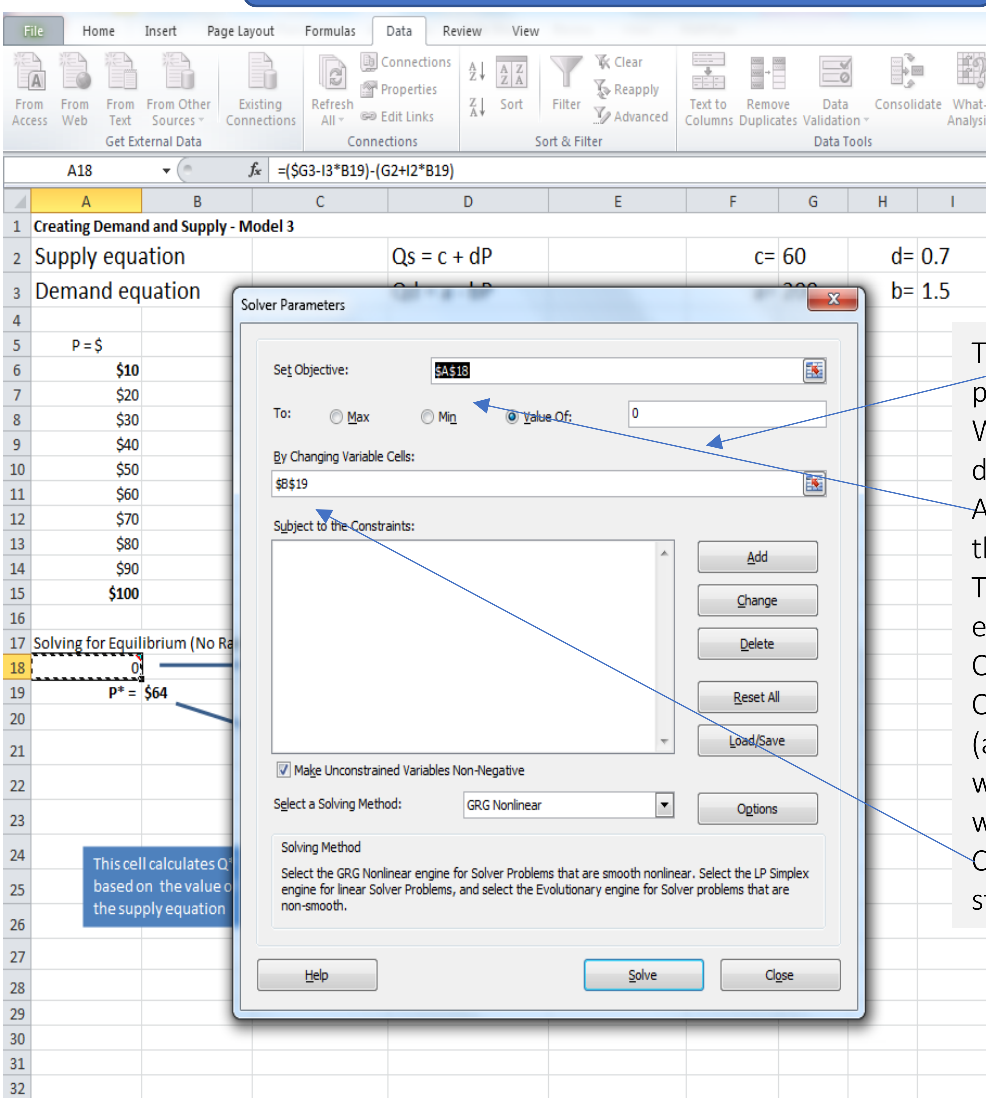
$$Q_D = a_0 + a_1 P$$
$$Q_S = b_0 + b_1 P$$
$$Q_D = Q_S$$

The last equality is an essential element of the model.

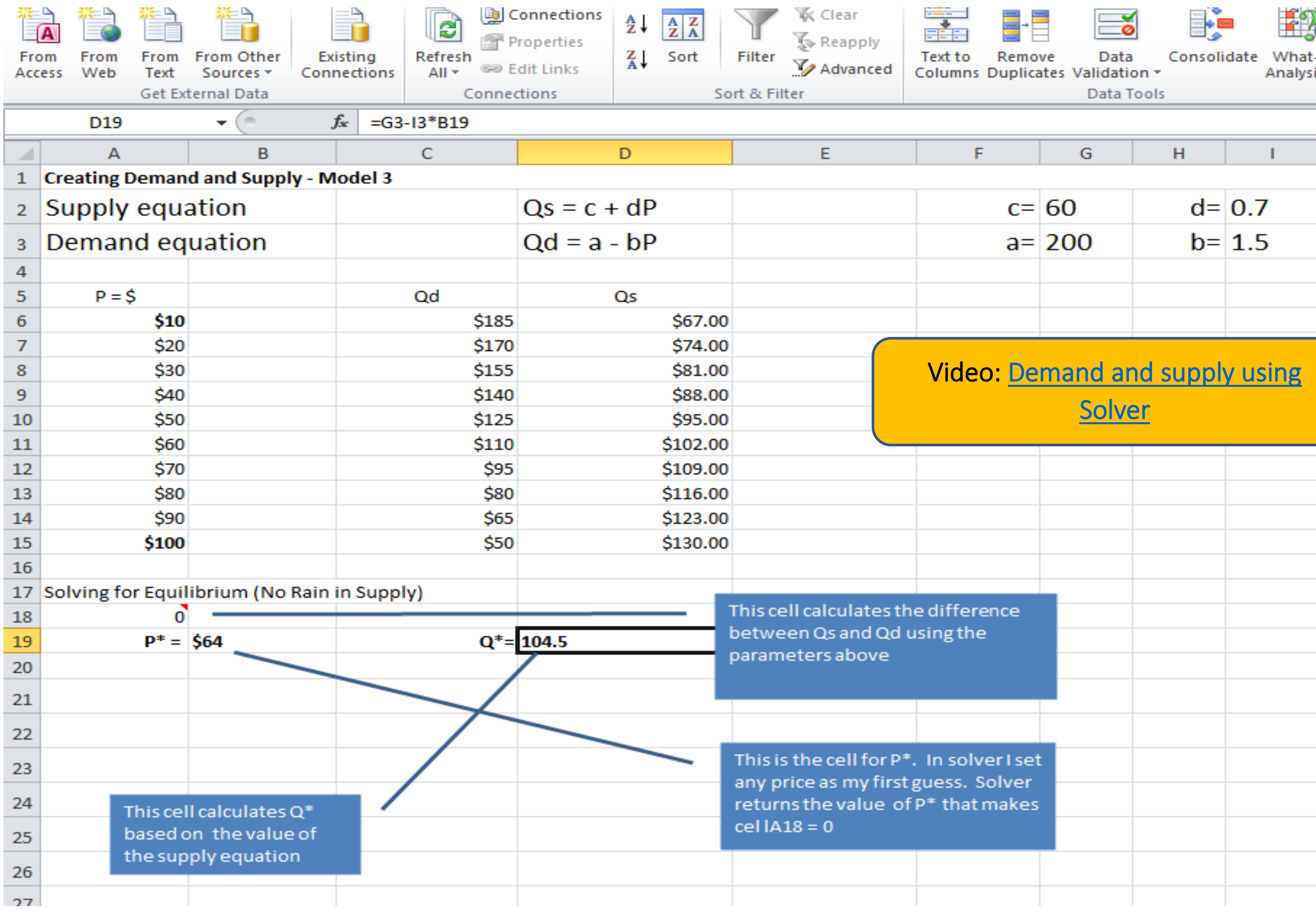


This is a standard supply and demand problem. Cells D2 and D3 are not Excel formulas; these are only text describing the supply and demand equations. Qs and Qd are dependent variables, and P is the independent variable. The parameters are a, b, c, and d. Cells F2:F3 and H2:H3 are labels; cells G2:G3 and I2:I3 have values.

Demand and Supply Equilibrium Using Solver.xlsx



The objective is to find the point where Qs = Qd.
We write the equations for demand minus supply in cell A18, which is the formula in the equation window.
The key here is to realize that equilibrium exists when Qd = Qs, which is the same as Qd-Qs= 0, which is the same as (a+bP) –(C+dP) = 0, which is what appears in the formula window for Cell A18.
Cell B19 has our “guess” to start Solver.



Video: Demand and supply using Solver

Problem

- Create a linear demand and supply function with the parameters of c=75, d=1.2, a=35, and b=1.
- Find the equilibrium.
- Now change the demand equation to

  $Q_d = c - d*P^{1.2}$

- Caution 1.... You need to figure out how to program this new demand (it is non-linear).
- Caution 2.... The objective function needs to reflect this new demand relation.

3. Summary

Module 10 introduces Excel's Solver as a general mechanism for finding optimum values for maximization, minimization, and equilibrium problems. The structure of many optimization problems remains constant, comprising of three elements:

1. Objective function (what is being maximized/minimized/equated). This will be an equation, which for Excel will be a formula in a specific cell. Creating this formula is usually the tricky part.
2. Constraints, such as explicit statements about the budget or costs, and implicit constraint, the most common being that all solutions much be positive values.
3. Policy parameters are the variables that vary to find the solution. This may be price, interest, or whatever.

The key to optimization is being able to recognize these components in general problems, create the equations that express the three components, create the Excel structure, and then finally translate this into the Solver dashboard.

Annex: Excel functions

This Module introduced no additional Excel formulas and functions. The important skill is mastering Solver.